11.5.

We already know that $H \cap K$ is a subgroup of H; it remains to show that $H \cap K$ is normal in H. It suffices to show that $h(H \cap K)h^{-1} \subseteq H \cap K$ for any $h \in H$. Fix $h \in H$. Since $gKg^{-1} = K$ for all $g \in G$, a fortiori $hKh^{-1} = K$. Thus $h(H \cap K)h^{-1} \subseteq K$. Note $h(H \cap K)h^{-1} \subseteq H$ since H is closed under multiplication. THus $h(H \cap K)h^{-1} \subseteq H \cap K$ as desired.

11.7.

Suppose $x \in H$ and $y \in K$. Consider $z = xyx^{-1}y^{-1}$. Since K is normal, $xyx^{-1} \in K$ and so $z = (xyx^{-1})\,y^{-1} \in K$. Since H is normal, $yx^{-1}y^{-1} \in H$ and so $z = x\,(yx^{-1}y^{-1}) \in K$. Thus $z \in H \cap K$ and so $e = z$. Thus $xyx^{-1}y^{-1} = e$ or $xy = yx$.

11.8.

Since N and H are subgroups, $e \in NH$ and NH is non-empty. Suppose $x, y \in NH$. Then there exist $n, n' \in N$ and $h, h' \in H$ such that $x = nh$ and $y = n'h'$. Since $N \lhd G$, we know $hN = Nh$. Thus there exists $n'' \in N$ such that $hn' = n''h$. Thus $xy = nhn'h' = nn''hh' \in NH$ and NH is closed under multiplication. Since $N \lhd G$, there exists $n''' \in N$ such that $h^{-1}n^{-1} = n'''h$. Thus $x^{-1} = (nh)^{-1} = n'''h \in NH$, and NH is closed under inverses. We conclude NH is a subgroup of G.

12.4.

(a) The group $\mathbb{Z}_{12}$ is finite, but $\mathbb{Q}^+$ is infinite. Therefore they are not isomorphic.

(b) (Corrected Nov. 13.) The groups are isomorphic. Consider the function $f : 2\mathbb{Z} \to 3\mathbb{Z}$ defined by $f(x) = \frac{3}{2}x$. The function is well-defined since $\frac{3}{2}$ times an even number is divisible by 3. For $x, y \in 2\mathbb{Z}$ we have $f(x+y) = \frac{3}{2}(x+y) = \frac{3}{2}x + \frac{3}{2}y = f(x) + f(y)$, so f is a homomorphism. The function is a bijection since the inverse function can be described as $f^{-1} : 3\mathbb{Z} \to 2\mathbb{Z}$ with $f^{-1}(x) = \frac{2}{3}x$.

(c) The group $\mathbb{R} - \{0\}$ contains -1, which has order 2. The group $\mathbb{R}$ has no elements of order 2. Therefore they are not isomorphic.

(d) The groups are isomorphic. Consider the function $f : V \to \mathbb{Z}_2 \times \mathbb{Z}_2$ given by $f(e) = (0,0)$, $f(a) = (1,0)$, $f(b) = (0,1)$ and $f(c) = (1,1)$. This is a bijection. To see that it is a group homomorphism, we compare the full multiplication tables:

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$+$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

(e) Every element of $\mathbb{Z}_3 \times \mathbb{Z}_3$ has order at most 3. However, $\mathbb{Z}_9$ has an element of order 9. Therefore they are not isomorphic.

(f) The groups are isomorphic. Consider the function $f : \mathbb{R}^+ \times \mathbb{Z}_2 \to \mathbb{R} - \{0\}$ via

$$f(a,b) = \begin{cases} a & \text{if } b = 0 \\ -a & \text{if } b = 1. \end{cases}$$

This is a well-defined bijective function. We check that it is a homomorphism by considering four cases:

$$\begin{aligned}
f((x,0)*(y,0)) &= f((xy,0)) = xy = xy = f((x,0))f(y,0)) \\
f((x,0)*(y,1)) &= f((xy,1)) = -xy = x(-y) = f((x,0))f(y,1)) \\
f((x,1)*(y,0)) &= f((xy,1)) = -xy = (-x)y = f((x,1))f(y,0)) \\
f((x,1)*(y,1)) &= f((xy,0)) = xy = (-x)(-y) = f((x,1))f(y,1))
\end{aligned}$$

12.11.

(a) Let $H = \langle(1,0)\rangle$ and $K = \langle(0,2)\rangle$. Since H and K are both cyclic groups of order 2, then $H \cong K$. Now G/H is generated by $(0,1)+H$ and is thus cyclic of order 4; in particular there is exactly one element of order 2 in G/H. However, $(1,0)+K$ and $(0,1)+K$ are non-equal elements of order 2 in G/K. We conclude that $G/H \ncong G/K$.

(b)

$$A = \{(0,0),(1,0),(0,2),(1,2)\}, \quad B = \{(0,1),(0,2),(0,3),(0,4)\}.$$

Note that A has 3 elements of order 2 while B only has one element of order 2. Thus $A \ncong B$. However, note that G/A and G/B both have $8/4 = 2$ elements. All groups of order 2 are isomorphic and we conclude that $G/A \cong G/B$.