11.1.

Let h be an element of $\mathrm{SL}(2,\mathbb{R})$ and g be an element of $\mathrm{GL}(2,\mathbb{R})$. We have $\det(ghg^{-1}) = \det(g)\det(h)\det(g)^{-1} =$ $\det(h) = 1$. Therefore $ghg^{-1} \in \mathrm{SL}(2,\mathbb{R})$. We conclude that $\mathrm{SL}(2,\mathbb{R})$ is normal in $\mathrm{GL}(2,\mathbb{R})$.

11.3.

Since $|H| = 2$, we know $H = \{e, x\}$ for some $e \neq x \in G$. Since H is normal, $yH = Hy$ for all $y \in G$. Now $yx \neq ey$ since $x \neq e$. Thus $yx = xy$ for all $y \in G$. We conclude that $x \in Z(G)$. Thus $H = \langle x \rangle \subseteq Z(G)$.

11.4.

Recall that $H \cap K$ is a subgroup of G, so it remains only to show that it is normal. It suffices to show that $g(H \cap K)g^{-1} \subseteq H \cap K$ for any $g \in G$. Suppose $x \in g(H \cap K)g^{-1}$. Then $x = gyg^{-1}$ where $y \in H \cap K$. In particular $y \in H$, so $x = gyg^{-1} \in gHg^{-1} = H$ since H is normal. Similarly, $x \in K$. Thus $x \in H \cap K$ as desired.

11.14.

Let x be the element $H + (5,8)$ in G/H. Note that

$$H = \{(0,0), (2,2), (4,4), (6,6), (8,8), (10,10)\}.$$

Computing multiples we find:

$$\begin{aligned} 2x &= H + (10,16) = H + (10,4) \neq H \\ 3x &= H + (15,12) = H + (3,0) \neq H \\ 4x &= H + (8,8) = H + (8,8) = H \end{aligned}$$

and conclude that $H + (5,8)$ has order 4.

12.1.

(a) (Corrected Nov. 5.) Indeed $\phi(xy) = |xy| = |x||y| = \phi(x)\phi(y)$ is a homomorphism. Every element x in $\mathbb{R}^+$ satisfies $\phi(x) = |x| = x$, so ϕ is surjective (an epimorphism). However, $\phi(1) = \phi(-1)$ and $1 \neq -1$, so ϕ is not a monomorphism.

(b) Note that $\phi(xy) = \sqrt{xy} = \sqrt{x}\sqrt{y} = \phi(x)\phi(y)$ for all positive real numbers x, y. Every positive real number is the square root of a positive real number, so ϕ is surjective (an epimorphism). Since 1 is the unique positive real with square root 1, we see that $\ker(\phi) = \{1\}$, so ϕ is injective (a monomorphism). Thus ϕ is an isomorphism.

(c) Given two polynomials p, q, we see that $\phi(p+q) = (p+q)(1) = p(1) + q(1) = \phi(p) + \phi(q)$, so ϕ is a homomorphism. For all $k \in R$, the polynomial $p(x) = k$ satisfies $\phi(p) = k$ so ϕ is surjective (an epimorphism). However, $\phi(p) = 1$ for both $p(x) = 1$ and $p(x) = x$, so ϕ is not injective. Thus ϕ is not a monomorphism nor an isomorphism.

(d) Given two polynomials p, q, the sum rule states that $\phi(p+q) = (p+q)' = p' + q' = \phi(p) + \phi(q)$, so ϕ is a homomorphism. Any polynomial has a polynomial antiderivative, so ϕ is surjective and thus an epimorphism. However, $\phi(p) = 0$ for any constant polynomial p. Thus ϕ is not a monomorphism nor an isomorphism.

(e) Recall that the empty set is the identity of G. Since $\phi(\emptyset) = A \triangle \emptyset = A \neq \emptyset$, we conclude ϕ is not a homomorphism.

12.2.

Suppose G is abelian. The function $\phi : G \to G$ is a bijection since it has an inverse $\phi^{-1} = \phi$. Let $x, y \in G$. Then $\phi(xy) = (xy)^{-1} = y^{-1}x^{-1} = \phi(y)\phi(x)$. Since G is abelian, $\phi(xy) = \phi(x)\phi(y)$.

Now suppose G is not abelian. Then there exist $x, y \in G$ such that $xy \neq yx$. Taking inverses of both sides we see that $y^{-1}x^{-1} \neq x^{-1}y^{-1}$. Now observe that $\phi(xy) = (xy)^{-1} = y^{-1}x^{-1}$ while $\phi(x)\phi(y) = x^{-1}y^{-1}$. Thus $\phi(xy) \neq \phi(x)\phi(y)$ and we conclude that ϕ is not an isomorphism.

12.14.

First, we will show that $Z(H) \subseteq \phi(Z(G))$. Suppose $z \in Z(H)$. Since ϕ is surjective, there exists an element $w \in G$ such that $\phi(w) = z$. Let g be an arbitrary element of G. Then

$$\phi(gw) = \phi(g)\phi(w) = \phi(g)z = z\phi(g) = \phi(w)\phi(g) = \phi(wg).$$

Since ϕ is injective, $\phi(gw) = \phi(wg)$ implies $gw = wg$. Thus $w \in Z(G)$. Thus $z = \phi(w) \in \phi(Z(G))$.

Now, we show that $\phi(Z(G)) \subseteq Z(H)$. Suppose $z \in Z(G)$. Let h be an arbitrary element of H. Since ϕ is surjective, there exists an element $g \in G$ such that $\phi(g) = h$. Since $z \in Z(G)$, we have $zg = gz$. Thus $\phi(zg) = \phi(gz)$. Since ϕ is a homomorphism, we have $\phi(z)\phi(g) = \phi(g)\phi(z)$. Thus $\phi(z)h = h\phi(z)$. Since h was arbitrary, we conclude that $\phi(z) \in Z(H)$.

We conclude that $\phi(Z(G)) = Z(H)$. Since ϕ is an isomorphism, by restricting the domain and codomain we obtain a bijection $\psi : Z(G) \to Z(H)$. The function ψ is a homomorphism since ϕ is a homomorphism.