8.26(a).

Let $h \in S_n$ be a permutation and let $c = (x_1 \cdots x_r)$. We will evaluate $g = h \circ c \circ h^{-1}$ on every element of $\{1, \ldots, n\}$. First, we evaluate g on $h(x_i)$ for some index $1 \leq i < r$. We find

$$g(h(x_i) = h(c(h^{-1}(h(x_i)))) = h(c(x_i)) = h(x_{i+1}).$$

Now We find

$$g(h(x_r) = h(c(h^{-1}(h(x_r)))) = h(c(x_r)) = h(x_1).$$

Now, suppose y is not of the form $h(x_i)$ for any x_i. We find

$$g(y) = h(c(h^{-1}(y))) = h(h^{-1}(y)) = y.$$

Thus

$$g(y) = \begin{cases} h(x_{i+1}) & y = h(x_i), 1 \leq i < r \\ h(x_1) & y = h(x_r) \\ y & \text{otherwise.} \end{cases}$$

9.7.

$$\begin{aligned} H + (0,0) &= \{(0,0), (1,0), (2,0)\}, \\ H + (0,1) &= \{(0,1), (1,1), (2,1)\}, \\ H + (0,2) &= \{(0,2), (1,2), (2,2)\} \end{aligned}$$

9.8.

$$\begin{aligned} H + (0,0) &= \{(0,0), (0,2)\}, & H + (0,1) &= \{(0,1), (0,3)\}, \\ H + (1,0) &= \{(1,0), (0,2)\}, & H + (1,1) &= \{(1,1), (0,3)\}, \\ H + (2,0) &= \{(2,0), (2,2)\}, & H + (2,1) &= \{(2,1), (2,3)\}, \\ H + (3,0) &= \{(3,0), (3,2)\}, & H + (3,1) &= \{(3,1), (3,3)\} \end{aligned}$$

9.13.

First, we show $\sim$ is reflexive. Suppose $x \in G$. Since A and B are subgroups, $e \in A$ and $e \in B$. Since $x = exe$, we conclude $x \sim x$ as desired.

Next, we show $\sim$ is symmetric. Suppose $x, y \in G$ and $x \sim y$. Then $x = ayb$ where $a \in A$ and $b \in B$. Then $y = a^{-1}xb^{-1}$, where $a^{-1} \in A$ and $b^{-1} \in B$. Thus $y \sim x$ as desired.

Finally, we show $\sim$ is transitive. Suppose $x, y, z \in G$, $x \sim y$ and $y \sim z$. Then $x = ayb$ and $y = a'zb'$ for $a, a' \in G$ and $b, b' \in G$. Thus $x = a(a'zb')b = (aa')z(bb')$, where $aa' \in A$ and $bb' \in B$. Thus $x \sim z$ as desired.

We conlude $\sim$ is an equivalence relation since it is reflexive, symmetric, and transitive.

10.2(c).

In $\mathbb{Z}_{112}$, we find $o(100) = 112/\gcd(112, 100) = 112/4 = 28$. Now $|H| = o(100) = 28$, since 100 generates the group H. Since G is finite, $[G : H] = |G|/|H| = 112/28 = 4$.

10.3(b).

Note that the first $\langle 2\rangle$ is a subgroup of $\mathbb{Z}_6$ while the second is a subgroup of $\mathbb{Z}_4$. We use the $\bar{2}_6$ and $\bar{2}_4$ notation to reduce confusion. Observe that $\langle \bar{2}_6\rangle = \{\bar{0}_6, \bar{2}_6, \bar{4}_6\}$ as a subgroup of $\mathbb{Z}_6$, but $\langle \bar{2}_4\rangle = \{\bar{0}_4, \bar{2}_4\}$ as a subgroup of $\mathbb{Z}_4$. Thus $|H| = |\langle \bar{2}_4\rangle||\langle \bar{2}_6\rangle| = 3 \cdot 2 = 6$. Now $|G| = |\mathbb{Z}_6||\mathbb{Z}_4| = 6 \cdot 4 = 24$. Since G is finite, $[G : H] = |G|/|H| = 24/6 = 4$.

10.5.

By Lagrange's theorem, every subgroup of G has order dividing 8. Suppose there exists an element x of order 8, then $\langle x\rangle \subseteq G$. Since $|\langle x\rangle| = o(x) = 8$, we conclude that $G = \langle x\rangle$ and thus G is cyclic — a contradiction. Thus every element a of G has order 1, 2 or 4. Since 1 and 2 divide 4, we conclude that $a^4 = e$ for every $a \in G$.

10.6.

Let x be an element in $H \cap K$. Thus $\langle x\rangle$ is a subgroup of both H and K. By Lagrange's theorem, $|\langle x\rangle|$ divides both $|H|$ and $|K|$. Since $\gcd(12, 5) = 1$, we conclude that $|\langle x\rangle| = 1$. Thus $x = e$ and so $H \cap K = \{e\}$.