6.1.

Recall that the order of m in $\mathbb{Z}_n$ is $n/\gcd(n,m)$. Thus, we use Theorem 6.1(i) to verify every case.

a)

$$o(4,9) = \operatorname{lcm}(o(\overline{4}_{18}), o(\overline{9}_{18})) = \operatorname{lcm}\left(\frac{18}{\gcd(18,4)}, \frac{18}{\gcd(18,9)}\right) = \operatorname{lcm}\left(\frac{18}{2}, \frac{18}{9}\right) = \operatorname{lcm}(9,2) = 18.$$

b)

$$o(7,5) = \operatorname{lcm}\left(\frac{12}{\gcd(12,7)}, \frac{8}{\gcd(8,5)}\right) = \operatorname{lcm}\left(\frac{12}{1}, \frac{8}{1}\right) = 24.$$

c)

$$o(8,6,4) = \operatorname{lcm}\left(\frac{18}{\gcd(18,8)}, \frac{9}{\gcd(9,6)}, \frac{8}{\gcd(8,4)}\right) = \operatorname{lcm}\left(\frac{18}{2}, \frac{9}{3}, \frac{8}{4}\right) = 18.$$

d)

$$o(8,6,4) = \operatorname{lcm}\left(\frac{9}{\gcd(9,8)}, \frac{17}{\gcd(17,6)}, \frac{10}{\gcd(10,4)}\right) = \operatorname{lcm}\left(\frac{9}{1}, \frac{17}{1}, \frac{10}{2}\right) = 765.$$

6.2.

From Theorem 6.1 of Saracino, a product of cyclic groups is cyclic if and only if their orders are pairwise coprime .

a) Since $\gcd(12,9) = 3 \neq 1$, the group $\mathbb{Z}_{12} \times \mathbb{Z}_9$ is not cyclic.

b) Since $\gcd(10,85) = 5 \neq 1$, the group $\mathbb{Z}_{10} \times \mathbb{Z}_{85}$ is not cyclic.

c) Since $\gcd(4,6) = 2 \neq 1$, the group $\mathbb{Z}_4 \times \mathbb{Z}_{25} \times Z_6$ is not cyclic.

d) Note that $\gcd(22,21) = 1$, $\gcd(22,65) = 1$, and $\gcd(21,65) = 1$. Thus the group $\mathbb{Z}_{22} \times \mathbb{Z}_{21} \times Z_{65}$ is cyclic.

6.9.

We need to prove an equality of sets. Suppose $z \in Z(G_1 \times \cdots \times G_n)$. Write $z = (z_1, \ldots, z_n)$ where $z_i \in G_i$ for each $i \in \{1, \ldots, n\}$. Fix an index i. Suppose $g \in G_i$. Consider $h = (e, \ldots, e, g, e, \ldots, e)$, where g is in the ith position and the identity is in every other position. Since $z \in Z(G)$, we have $zh = hz$. Looking at the ith position, this implies $z_i g = g z_i$. Since this holds for every $g \in G_i$, we conclude that $z_i \in Z(G_i)$. Thus $z \in Z(G_1) \times \cdots \times Z(G_n)$.

Now we consider the other direction. Consider $z \in Z(G_1) \times \cdots \times Z(G_n)$. Let $g \in G_1 \times \cdots \times G_n$. Write $z = (z_1, \ldots, z_n)$ and $g = (g_1, \ldots, g_n)$. We have $z_i g_i = g_i z_i$ for every index i since $z_i \in Z(G_i)$. Thus

$$zg = (z_1 g_1, \ldots, z_n g_n) = (g_1 z_1, \ldots, g_n z_n) = zg.$$

Since g was arbitrary, we conclude that $z \in Z(G_1 \times \cdots G_n)$.

8.1.

a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 6 & 3 & 5 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix}$$

8.2.

a) As a product of disjoint cycles: (13)(265). As a product of transpositions: (13)(25)(26). Since there are three transpositions, the permutation is odd.

b) As a product of disjoint cycles: (124)(365). As a product of transpositions: (14)(12)(35)(36). Since there are 4 transpositions, the permutation is even.

c) As a product of disjoint cycles: (15)(26). As a product of transpositions: (15)(26). Since there are 2 transpositions, the permutation is even.

d) As a product of disjoint cycles: (1634)(25). As a product of transpositions: (14)(13)(16)(25). Since there are 4 transpositions, the permutation is even.

8.7.

The elements $a = (12)$ and $b = (23)$ are in every symmetric group S_n when $n \geq 3$. We compute $ab = (123)$ and $ba = (132)$. Thus $ab \neq ba$ and we conclude S_n is non-abelian.

8.11.

a) Consider $x = (12345)$ and $y = (56789)$. One checks that $xy = (123456789)$. Since x and y are 5-cycles, they have order 5. Since xy is a 9-cycle, $o(xy) = 9$.

b) A permutation σ in S_9 has a decomposition as disjoint cycles of orders $r_1, \ldots, r_n$ where $9 = r_1 + \cdots + r_n$. (Note that we allow "1-cycles" which are simply the trivial element in S_9.) The order of σ is $\operatorname{lcm}(r_1, \ldots, r_n)$. We claim that the largest order is 20 obtained from $9 = 5 + 4$. Indeed, the lcm of integers $1, 2, 3, 4$ is 12, so we only need to consider permutations with a cycle of order 5 or more. The remaining possible cycle types are

$$9,\ 8+1,\ 7+2,\ 7+1+1,\ 6+3,\ 6+2+1,\ 6+1+1,$$
$$5+3+1,\ 5+2+2,\ 5+2+1+1,\ 5+1+1+1+1$$

which all have lower lcm.