5.1.

We use Theorem 5.1 to verify every case.

a) Yes. A sum of rational numbers is rational and the negation of a rational number is rational.

b) Yes. A sum of integers is integral and the negation of an integer is integral.

c) No. $3 \in \mathbb{Z}^+$ but $-3 \notin \mathbb{Z}^+$.

d) Yes. A product of positive rational numbers is a positive rational number and the inverse of a positive rational number is a positive rational number.

e) No. $2, 4 \in H$ but $2 \oplus 4 = 6 \notin H$.

f) Yes. $(a, -a) * (b, -b) = (a + b, -(a + b))$ and $(a, -a)^{-1} = (-a, a)$. for all $a, b \in \mathbb{R}$.

g) No. $I^2 \notin Q_8$.

h) Yes. One verifies that $X \triangle Y \in G$ for all $X, Y \in H$. Every element of H is its own inverse (and thus in H).

i) Yes. Let $A, B \subseteq Y$. Then $A \triangle B$ is also a subset of Y. Moreover, since $A \triangle A = \emptyset$, every element of H is its own inverse.

5.4.

We use Corollary 5.6 from the text.

a) The distinct positive divisors of $\mathbb{Z}_{18}$ are 1, 2, 3, 6, 9, and 18. Thus, there are exactly 6 subgroups of $\mathbb{Z}_{18}$ and they are $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$, $\langle 6 \rangle$, $\langle 9 \rangle$, and $\langle 0 \rangle$.

b) The distinct positive divisors of $\mathbb{Z}_{35}$ are 1, 5, 7, 35. Thus, there are exactly 6 subgroups of $\mathbb{Z}_{35}$ and they are $\langle 1 \rangle$, $\langle 5 \rangle$, $\langle 7 \rangle$, and $\langle 0 \rangle$.

c) The distinct positive divisors of $\mathbb{Z}_{36}$ are 1, 2, 3, 4, 6, 9, 12, 18 and 36. Thus, there are exactly 6 subgroups of $\mathbb{Z}_{36}$ and they are $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$, $\langle 4 \rangle$, $\langle 6 \rangle$, $\langle 9 \rangle$, $\langle 12 \rangle$, $\langle 18 \rangle$, and $\langle 36 \rangle$.

5.7.

Suppose x^m is a generator of G. Since $x \in G$, there must exist $k \in \mathbb{Z}$ such that $x = (x^m)^k$. Since x has order n, this means that $1 \equiv mk \mod n$. Thus there exists $\ell \in \mathbb{Z}$ such that $1 = mk + n\ell$. By Bezout's Lemma, this means $\gcd(m, n) = 1$.

Now suppose $\gcd(m, n) = 1$. By Bezout's Lemma, this means that there exist $a, b \in \mathbb{Z}$ such that $1 = ma + nb$. Thus, $1 \equiv ma \mod n$. Thus $x = x^1 = x^{ma} = (x^m)^a$. Since $G = \langle x \rangle$, every element y of G can be written in the form $y = x^k$ for some integer k. Thus $y = (x^m)^{ak}$. We conclude x^m is a generator for G

5.14.

Let H and K be subgroups of G. We will apply Theorem 5.1 to show $H \cap K$ is a group. First, since $e \in H$ and $e \in K$, we conclude that $e \in H \cap K$. Thus $H \cap K$ is nonempty. Next, suppose $a, b \in H \cap K$. Thus $a, b \in H$ and so $ab \in H$ since H is a subgroup. Similarly, $ab \in K$. Thus $ab \in H \cap K$. Finally, suppose $a \in H \cap K$. Since $a \in H$ and H is a subgroup, we have $a^{-1} \in H$. Similarly, $a^{-1} \in K$. Thus $a^{-1} \in H \cap K$. Thus $H \cap K$ is a group by Theorem 5.1.

5.22.

Recall that $Z(G)$ is the center of the group G. By definition, $z \in Z(G)$ if and only if $zx = xz$ for all $x \in G$. We will apply Theorem 5.1 to show $Z(G)$ is a group. First, observe that $e \in Z(G)$ since $ex = x = xe$ for all $x \in G$. Thus $Z(G)$ is nonempty. Next, suppose $a, b \in Z(G)$. Consider $x \in G$. We have $ax = xa$ and $bx = xb$, thus $abx = axb = xab$. Since x was arbitrary, we conclude $ab \in Z(G)$. Finally, suppose $a \in Z(G)$ and let $x \in G$. We have $ax = xa$. Multiplying by a^{-1} on the left, we obtain $x = a^{-1}xa$. Multiplying by a^{-1} on the right, we obtain $xa^{-1} = a^{-1}x$. Since x was arbitrary, we conclude $a^{-1} \in Z(G)$. Thus $Z(G)$ is a group by Theorem 5.1.