2.4.

a)

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

b)

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

c)

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

2.6. No, $(S, *)$ is not a group. Observe that $(b * c) * c = c * c = a$ while $b * (c * c) = b * a = b$. Thus $*$ is not associative.

2.10. First we verify the multiplication is well-defined. For $a, b, c, d \in \mathbb{R}$ we compute

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{pmatrix}.$$

If we assume $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$, then

$$(ac - bd)^2 + (ad + bc)^2 = a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 = (a^2 + b^2)(c^2 + d^2) \neq 0.$$

Thus matrix multiplication gives a well-defined binary operation on G.

We know that multiplication is associative for all matrices, so the restriction to G must also be associative.

The identity matrix is an identity for G.

Observe that

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

so every element of G has an inverse in G.

Alternative: Some of the computations above can be avoided by observing that a^2+b^2 is the determinant of the matrix.

3.1. Observe that 10 is the inverse to 2 and 5 is the inverse to 7 in $(\mathbb{Z}_{12}, \oplus)$. Thus

$$\begin{aligned} 2 \oplus x \oplus 7 &= 1 \\ 10 \oplus 2 \oplus x \oplus 7 &= 10 \oplus 1 \\ x \oplus 7 &= 11 \\ x \oplus 7 \oplus 5 &= 11 \oplus 5 \\ x &= 4 \end{aligned}$$

3.2. Note that $A \triangle A = \emptyset$. Thus $A \triangle x = B$ iff $A \triangle A \triangle x = A \triangle B$ iff $x = A \triangle B$. Thus $x = \{1, 2, 5, 6, 7, 8\}$.

3.5. $(x * y * z)^{-1} = (y * z)^{-1} * x^{-1} = z^{-1} * y^{-1} * x^{-1}$

3.11. Let $a, b \in G$. We know $a * a = e$, $b * b = e$, and $(a * b) * (a * b) = e$. Thus

$$\begin{aligned} a * b * a * b &= e \\ a * (a * b * a * b) &= a * e \\ b * a * b &= a \\ (b * a * b) * b &= a * b \\ b * a &= a * b. \end{aligned}$$

Thus $(G, *)$ is abelian.

3.12. Since cancellation laws hold, the following are equivalences:

$$\begin{aligned} (x * y)^2 &= x^2 * y^2 \\ x * y * x * y &= x * x * y * y \\ y * x * y &= x * y * y \\ y * x &= x * y. \end{aligned}$$

Thus $(G, *)$ is abelian.