1.3.

a) Yes. $a + b^2$ is an integer whenever a and b are integers.

b) Yes. a^2b^3 is an integer whenever a and b are integers.

c) No. $\frac{a}{a^2+b^2}$ is undefined when $a = b = 0$.

d) No. $\frac{a^2+2ab+b^2}{a+b}$ is undefined when $a = 1$ and $b = -1$.

e) Yes. $a + b - ab$ is an integer whenever a and b are integers.

f) Yes. b is real whenever a and b are real.

g) No. $(1) * (-4) = |4| = 4$ is not in the set S.

h) No. $(6) * (6) = 36$ is not in the set S.

i) Yes. $a * b$ is in S whenever a and b are in S.

i) Yes. $(A \triangle B) \triangle B$ is a subset of X whenever A and B are subsets of X.

1.7.

For all sets A, B, we have $A \triangle B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \triangle A$. Thus symmetric difference is a commutative operation.

1.8.

The book has already established that

$$(A \triangle B) \triangle C \subseteq A \triangle (B \triangle C) \tag{1}$$

We need to prove the other containment $\supseteq$.

From the previous problem, we know symmetric difference is commutative. Thus

$$A \triangle (B \triangle C) = (B \triangle C) \triangle A = (C \triangle B) \triangle A.$$

Now, using equation (1) we already know, we obtain

$$(C \triangle B) \triangle A \subseteq C \triangle (B \triangle A).$$

Continuing using commutativity again, we have

$$C \triangle (B \triangle A) = (B \triangle A) \triangle C = (A \triangle B) \triangle C.$$

Thus we obtain

$$A \triangle (B \triangle C) \subseteq (A \triangle B) \triangle C$$

as desired.

Alternative: Prove the other direction following the textbook's approach very closely by starting with an element $x \in (A \triangle B) \triangle C$ and showing that it is in $A \triangle (B \triangle C)$.

Alternative: Writing $\overline{A} = X - A$ for the complement, we see that $A \triangle B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$. Using distributivity of unions and intersections, along with De Morgan's Laws, we compute that

$$(A \triangle B) \triangle C = (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap C) = A \triangle (B \triangle C).$$

2.1.

a) No. No identity: there is no positive real number e such that $e+1=1$.

b) Yes. The binary operation is well-defined since the sum of two multiples of 3 is a multiple of 3. The operation is associative since addition is associative. The identity element is 0. The inverse of x is $-x$.

c) No. No identity: there is no non-zero real number e such that $(-1)*e=|-e|=-1$.

d) Yes. The binary operation is well-defined. The binary operation is associative since multiplication is associative. The identity element is 1. The inverse of 1 is 1, while the inverse of -1 is -1.

e) Yes. The binary operation is well-defined. The binary operation is associative since multiplication is associative. The identity element is 1. The inverse of $\frac{a}{b}$ is $\frac{b}{a}$, which is also positive with a rational square root.

f) No. The operation is not associative since $((0,0)*(0,0))*(0,1)=(0,-1)$ but $(0,0)*((0,0)*(0,1))=(0,1)$.

g) Yes. The binary operation is well-defined. The binary operation is associative since addition and multiplication are associative. The identity element is $(0,1)$. The inverse of (x,y) is $(-x,\frac{1}{y})$

h) Yes.

The binary operation is well-defined. Indeed, if $a*b=1$, then $a+b-ab=1$. Rearranging, this occurs if and only if $(a-1)(b-1)=0$. But this can only occur if $a=1$ or $b=1$, which are not in the group.

The binary operation is associative. Indeed, for all $a,b,c\in\mathbb{R}-\{1\}$:

$$(a*b)*c=(a+b-ab)*c=(a+b-ab)+c-(a+b-ab)c$$
$$=a+b-ab+c-ac-bc+abc=a+(b+c-bc)-a(b+c-bc)=a*(b*c)$$

The identity element is 0 since $a*0=a=0*a$ for all $a\in\mathbb{R}-\{1\}$.

The inverse of a is $\frac{a}{a-1}$. This is well defined since $a\neq 1$ and is in the group since $\frac{a}{a-1}\neq 1$ for any real a. We check that

$$a*\frac{a}{a-1}=a+\frac{a}{a-1}-a\left(\frac{a}{a-1}\right)=\frac{a^2-1+a-a^2}{a-1}=0$$

for all $a\neq 1$ (and similarly for $\frac{a}{a-1}*a$).

i) Yes.

The binary operation is well-defined.

The binary operation is associative. Indeed, for all $a,b,c\in\mathbb{Z}$:

$$(a*b)*c=(a+b-1)+c-1=a+(b+c-1)-1=a*(b*c)$$

The identity element is 1 since $a*1=a=1*a$ for all $a\in\mathbb{Z}$.

The inverse of a is $2-a$. Indeed, $a*(2-a)=a+(2-a)-1=1$ and $(2-a)*a=1$.