University of South Carolina

MATH 544-001

Midterm Examination 3

November 14, 2024

Closed book examination | Time: 75 minutes

Instructions:

Notes, books, computer, phones, calculators or other aids are **not** allowed. Please write on only one side of each page. If you need more space than is provided, then ask for extra paper from the proctor. Simplify your final answers. Full credit will not be awarded for insufficient accompanying work.

There are 30 total points available.

Problem 1. Find a basis for the null space of the following matrix:

$$\begin{pmatrix} 1 & 0 & 2 & 3 & 4 & 5 \\ 2 & 4 & 4 & 6 & 0 & 10 \\ 4 & 4 & 8 & 0 & 1 & 2 \end{pmatrix}$$

Solution: We find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 0 & -8 & 0 \\ 0 & 4 & 0 & -12 & -15 & -18 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -12 & -7 & -18 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & \frac{7}{12} & \frac{3}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & \frac{9}{4} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & \frac{7}{12} & \frac{3}{2} \end{pmatrix}$$

We construct a basis vector in the null space from each free variable:

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{9}{4} \\ 2 \\ 0 \\ -\frac{7}{12} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix} \right\}$$

Problem 2. Find a basis for the column space of the following matrix:

$$\begin{pmatrix} 4 & 2 & 2 & 6 & 8 & 4 \\ 2 & 0 & 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 & 0 \\ 1 & 1 & 2 & 6 & 0 & 1 \end{pmatrix}$$

Solution: We find the row echelon form

$$\begin{pmatrix} 4 & 2 & 2 & 6 & 8 & 4 \\ 2 & 0 & 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 & 0 \\ 1 & 1 & 2 & 6 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 6 & 0 & 1 \\ 0 & 1 & 0 & 0 & 4 & 0 \\ 0 & -4 & -6 & -18 & 8 & 0 \\ 0 & -2 & -3 & -9 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 6 & 0 & 1 \\ 0 & 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & -6 & -18 & 24 & 0 \\ 0 & 0 & -3 & -9 & 8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 6 & 0 & 1 \\ 0 & 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 9 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The first three columns are the pivot columns, thus a basis for the column space is

$$\left\{ \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Comments: Note that we only need to know where the leading terms are, so we do not need to compute the full reduced row echelon form. This saves a bit of time.

Problem 3. The set

$$\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\3\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\4\\5\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^4$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ where

$$\mathbf{x} = \begin{pmatrix}1\\1\\2\\5\end{pmatrix}.$$

Solution: Suppose the basis is $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$. We need to compute the coefficients of the unique expression

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4.$$

To do this, we row reduce the augmented system

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1\\ 2 & 1 & 4 & 1 & 1\\ 3 & 1 & 5 & 0 & 2\\ 0 & 0 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1\\ 0 & 1 & 0 & -1 & -1\\ 0 & 1 & -1 & -3 & -1\\ 0 & 0 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1\\ 0 & 1 & 0 & -1 & -1\\ 0 & 0 & -1 & -2 & 0\\ 0 & 0 & 1 & 1 & 5 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -9\\ 0 & 1 & 0 & -1 & -1\\ 0 & 0 & 0 & -1 & 5\\ 0 & 0 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -14\\ 0 & 1 & 0 & 0 & -6\\ 0 & 0 & 1 & 0 & 10\\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

Thus we find

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix}-14\\-6\\10\\-5\end{pmatrix}$$

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 4. The dimension of a vector space V is the number of elements in V.

Solution: False. The vector space $V = \mathbb{R}^2$ has infinitely many elements, but the dimension is 2.

Problem 5. If $\mathcal{B}$ is a basis of a vector space V, then span $\mathcal{B} = V$ and $\mathcal{B}$ is linearly independent.

Solution: True. This is the definition of a vector space.

Problem 6. If T is an invertible linear transformation, then the kernel and the range are isomorphic vector spaces.

Solution: False. The identity transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible. The kernel is $\{0\}$, while the range is $\mathbb{R}^2$. These are different dimensions and cannot be isomorphic.

Problem 7. If H is a subspace of a finite-dimensional vector space V, then $\dim(H) \leq \dim(V)$.

Solution: True. This one of the consequences of Theorem 12 in the text.

Problem 8. Every vector space has a unique basis.

Solution: False. The vector space $\mathbb{R}^2$ has many different bases. For example,

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ and } \mathcal{C} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

are distinct bases of $\mathbb{R}^2$.

Problem 9. Let $\mathbb{P}_8$ be the set of polynomials of degree at most 8. Let V be the subset of $\mathbb{P}_8$ consisting of polynomials f such that $f(4) = 0$. Prove that V is a subspace of $\mathbb{P}_8$.

Solution: We verify that V is a subspace.

First, the zero polynomial $z(x) = 0$ satisfies $z(4) = 0$ so $z \in V$.

Second, suppose $f, g \in V$. Then $f(4) = 0$ and $g(4) = 0$ by definition. Now $(f+g)(4) = f(4)+g(4) = 0$. Thus $f + g \in V$.

Finally, suppose $f \in V$ and $c \in \mathbb{R}$. Then $f(4)$ by definition. Thus $(cf)(4) = cf(4) = c \cdot 0 = 0$, Thus $cf \in V$.

Therefore V is a subspace.

Comments: Many students found an explicit basis for V, which was not necessary.

Problem 10. Let A be an $n \times n$ matrix such that $A^2 = 0$. Prove that the column space $\text{Col}(A)$ is a subset of the null space $\text{Nul}(A)$.

Solution: Suppose $\mathbf{x}$ is in $\text{Col}(A)$. Recall that

$$\text{Col}(A) = \{\mathbf{x} \mid \mathbf{x} = A\mathbf{y} \text{ for some } \mathbf{y} \text{ in } \mathbb{R}^n\}$$

This means that there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} = A\mathbf{y}$. Therefore $A\mathbf{x} = A(A\mathbf{y}) = A^2\mathbf{y} = 0\mathbf{y} = \mathbf{0}$. Since $A\mathbf{x} = \mathbf{0}$, we conclude that $\mathbf{x} \in \text{Nul}(A)$.

Comments: The matrix A does not have to be the zero matrix. For example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

both satisfy $A^2 = 0$. It is true that $\text{rank}(A) \leq \text{nullity}(A)$ if $A^2 = 0$, but this is weaker than what you are asked to prove.