Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

For each of the following, compute the matrix or indicate that the expression is undefined.

Problem 1. $\det(C)A + CA$

Solution:

$$\det(C)A + CA = (-2)\begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 7 & -3 & 7 \\ 10 & -4 & 12 \end{pmatrix} = \begin{pmatrix} 5 & -3 & -1 \\ 6 & -2 & 10 \end{pmatrix}$$

Problem 2. $2AB + C^{-1}$

Solution:

$$2AB + C^{-1} = (2)\begin{pmatrix} 1 & 6 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{27}{2} \\ 3 & \frac{19}{2} \end{pmatrix}$$

Problem 3. Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 2 \\ 2 & 10 & 6 \end{pmatrix}$$

or show that it does not exist.

Solution: We form an augmented matrix and row reduce as follows:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 2 & 10 & 6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & -1 & 1 & 0 \\ 0 & 8 & 6 & -2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{8}{3} & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 1 & -4 & \frac{3}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 1 & -4 & \frac{3}{2} \end{array}\right)$$

Thus the inverse is

$$\begin{pmatrix} 2 & -3 & 1 \\ -1 & 3 & -1 \\ 1 & -4 & \frac{3}{2} \end{pmatrix}.$$

Problem 4. Find the determinant of the following matrix:

$$\begin{pmatrix} 1 & 2 & 4 & 0 \\ 1 & 3 & 2 & 0 \\ 5 & 17 & 12 & 1 \\ 4 & 2 & 3 & 0 \end{pmatrix}$$

Solution: Using the cofactor expansion along the last column, we see that the determinant is equal to

$$(-1)(1)\det\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Now using cofactor across the first row, we find:

$$-\left[\det\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 2\det\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + 4\det\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\right] = -(9-4) + 2(3-8) - 4(2-12) = 25$$

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 5. Let A, B, C be $n \times n$ matrices. Then $A(B+C) = AB + AC$.

Solution: True. This is Theorem 2(b) from 2.1 in the Textbook.

Problem 6. Every function f can be written $f(x) = Ax$ for some matrix A.

Solution: False. First of all, this only applies to functions with domain and codomain of the form $\mathbb{R}^n$. Moreover, even then, the functions must be linear. For example, $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is not of the desired form.

Problem 7. Let A be an $n \times n$ matrix. The matrix A is invertible if and only if A^T is invertible.

Solution: True. This is (a) and (l) of the Invertible Matrix Theorem.

Problem 8. Let A be an $n \times n$ matrix. The matrix A is row equivalent to the $n \times n$ identity matrix if and only if the columns of A span $\mathbb{R}^n$.

Solution: True. This is (b) and (h) of the Invertible Matrix Theorem.

Problem 9. Let A, B be $n \times n$ matrices. Then $\det(A+B) = \det(A) + \det(B)$.

Solution: False. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Problem 10. Let A, B, C be $n \times n$ matrices. Suppose C is invertible and $AC = CB$. Prove that A is invertible if and only if B is invertible.

Solution: Suppose that A is invertible. Note that $B = C^{-1}AC$. Since a product of invertible matrices is invertible, B is invertible.

Conversely, suppose that B is invertible. Then $A = CBC^{-1}$ is a product of invertible matrices and is, therefore, invertible.

Comments: There are several variations on this argument that are valid. Note that if both BC and C are invertible, then B is indeed invertible, but you need more justification. Note that it is not always true that $A = B$ here; remember that matrix multiplication does not commute!

Problem 11. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ be surjective linear transformations. Prove that the composition $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is a surjective linear transformation.

Solution: For surjectivity, consider $\mathbf{x} \in \mathbb{R}^p$. Since S is surjective, there must exist $\mathbf{y} \in \mathbb{R}^m$ such that $S(\mathbf{y}) = \mathbf{x}$. Since T is surjective, there must exist $\mathbf{z} \in \mathbb{R}^n$ such that $T(\mathbf{z}) = \mathbf{y}$. Putting these together, we see that $(S \circ T)(\mathbf{z}) = \mathbf{x}$. Thus $S \circ T$ is surjective.

We saw in class that a composition of linear transformations is a linear transformation. To see this again, we check that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we have

$$(S \circ T)(\mathbf{v} + \mathbf{w}) = S(T(\mathbf{v} + \mathbf{w})) = S(T(\mathbf{v}) + T(\mathbf{w})) = S(T(\mathbf{v})) + S(T(\mathbf{w})) = (S \circ T)(\mathbf{v}) + (S \circ T)(\mathbf{w})$$

and

$$(S \circ T)(c\mathbf{v}) = S(T(c\mathbf{v}) = S(cT(\mathbf{v}) = cS(T(\mathbf{v}) = c(S \circ T)(\mathbf{v}).$$

Comments: I was only expecting a proof of surjectivity (onto-ness) and gave full credit for this. If you only proved linearity instead I gave some partial credit.
