**Problem 1.** Find the reduced row echelon form for the matrix

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 2 & 0 & 5 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

**Solution:**

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 2 & 0 & 5 \\ 3 & 2 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -4 & -1 \\ 3 & 2 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -4 & -1 \\ 0 & 2 & -5 & -5 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -4 & -1 \\ 0 & 0 & -1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -4 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 0 & 15 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & \frac{15}{2} \\ 0 & 0 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & \frac{15}{2} \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

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**Problem 2.** Describe all solutions of the following linear system.

$$\begin{aligned} x + 2y + 2w - v &= 1 \\ z + 3w + 5v &= 2 \\ u + v &= 7 \end{aligned}$$

**Solution:** The possible solutions have $y$, $w$ and $v$ as any real number and then the remaining variables are determined from

$$\begin{aligned} x &= 1 - 2y - 2w + v \\ z &= 2 - 3w - 5v \\ u &= 7 - v \end{aligned}$$

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**Problem 3.** Describe all solutions of $A\mathbf{x} = \mathbf{b}$ in parametric vector form, where

$$A = \begin{pmatrix} 1 & 0 & 2 & 4 & 0 & 4 \\ 0 & 1 & 3 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}.$$

**Solution:** Observe that the matrix $A$ is already in reduced row echelon form. If $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$, then the basic variables are $x_1, x_2, x_5$ and the free variables are $x_3, x_4, x_6$. Thus, the parametric vector form of the solution set is

$$\mathbf{x} = \begin{pmatrix} 5 \\ 3 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -4 \\ -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} -4 \\ -2 \\ 0 \\ 0 \\ -7 \\ 1 \end{pmatrix}$$

where $t, u, v \in \mathbb{R}$.

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**Problem 4.** Determine if the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}.$$

are linearly independent.

**Solution:** Let $A$ be the matrix with these vectors as columns. The vectors are linearly independent if and only if the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has no non-trivial solutions. By Gaussian elimination, we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 4 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is in echelon form and not every column has a pivot. Therefore there must be a non-trivial solution. Thus the vectors are linearly dependent.

(Note that $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = 0$ is an explicit solution, but we just need to show there exists a solution.)

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Determine whether each of the following statements are true or false. No justification is necessary.

**Problem 5.** Every linear system has a unique solution.

**Solution: False.** It is possible to have infinitely many or no solutions.

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**Problem 6.** If $A$ is an $n \times m$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^m$, then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$.

**Solution: True.** This is Theorem 5 (a) from §1.4 in the text.

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**Problem 7.** The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ are linearly dependent if and only if the equation $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ has a solution.

**Solution: False.** The given equation always has a solution; namely, the trivial one.

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**Problem 8.** If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not in the set spanned by the columns of $A$.

**Solution: True.** The span of the columns of $A$ is precisely the set of vectors that can be written as $A\mathbf{x}$ for some $\mathbf{x}$. Therefore, the equation $A\mathbf{x} = \mathbf{b}$ has no solution if and only if $\mathbf{b}$ is not in the span of the columns.

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**Problem 9.** Whenever a system has free variables, the solution set contains a unique solution.

**Solution: False.** A system with free variables may have zero or infinitely many solutions.

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**Problem 10.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. Show that, if $\mathbf{w}$ is in the set spanned by $\mathbf{v}$ and $\mathbf{u}$, then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent.

**Solution:** Suppose $\mathbf{w}$ is in the set spanned by $\mathbf{v}$ and $\mathbf{u}$. This means that there exist $a, b \in \mathbb{R}$ such that

$$\mathbf{w} = a\mathbf{v} + b\mathbf{u}.$$

Rearranging, we have

$$-b\mathbf{u} - a\mathbf{v} + \mathbf{w} = \mathbf{0}.$$

Since the coefficient of $\mathbf{w}$ is nonzero, we see that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent.

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