University of South Carolina
MATH 544-001

Final Examination - Sample B

December 12, 2024

Closed book examination Time: 150 minutes

Instructions:

Notes, books, computer, phones, calculators or other aids are **not** allowed. Please write on only one side of each page. If you need more space than is provided, then ask for extra paper from the proctor. Simplify your final answers. Full credit will not be awarded for insufficient accompanying work.

There are 60 total points available.

Problem 1. (5 points) Find the reduced row echelon form for the matrix

$$\begin{pmatrix} 1 & 4 & 2 & 0 & 4 \\ 2 & 8 & 4 & 1 & 3 \\ 0 & 1 & 3 & 2 & 1 \end{pmatrix}.$$

Problem 2. (5 points) Find a basis for the null space of the following matrix:

$$\begin{pmatrix} 2 & 3 & 0 & 1 & 5 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 4 & 1 & 1 & 8 \end{pmatrix}$$

Problem 3. (5 points) Consider the matrices

$$A = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 3 & 1 \\ -2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}.$$

For each of the following, compute the matrix or indicate that the expression is undefined.

AB

BA

$A - B$

$C^T + C^{-1}$

Problem 4. (5 points) Find the inverse of the matrix

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

or show that it does not exist.

Problem 5. (5 points) Find the determinant of the following matrix:

$$\begin{pmatrix} 5 & 6 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 & 1 \\ 4 & 5 & 1 & 0 & 7 \\ -2 & -1 & 3 & 3 & 3 \end{pmatrix}$$

Problem 6. (5 points) Compute the change-of-basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ where

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\} \text{ and } \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Problem 7. (5 points) If they exist, find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$ where

$$A = \begin{pmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{pmatrix}.$$

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 8. (1 point) Any finite linear combination of vectors in $\mathbb{R}^n$ can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector $\mathbf{x}$.

Problem 9. (1 point) A mapping $T : \mathbf{R}^n \to \mathbf{R}^m$ is injective if each vector in $\mathbf{R}^n$ maps onto a unique vector in $\mathbf{R}^m$.

Problem 10. (1 point) If $A\mathbf{x} = \mathbf{0}$ has the zero solution, then A must be the zero matrix.

Problem 11. (1 point) Let A, B be $n \times n$-matrices. If $AB = I_n$, then $B = A^{-1}$.

Problem 12. (1 point) If A is a square matrix, then $\det(A) = \det(A^T)$.

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 13. (1 point) Let A be an $n \times n$ matrix. Then $\det(2A) = 2^n \det(A)$.

Problem 14. (1 point) If A is a square matrix, then $\text{Col}(A) = \text{Nul}(A)$.

Problem 15. (1 point) If V is a finite dimensional vector space, then V is isomorphic to $\mathbb{R}^n$ for some non-negative integer n.

Problem 16. (1 point) Every square matrix has at least one real eigenvalue.

Problem 17. (1 point) Suppose A is a diagonalizable matrix. Every set of eigenvectors of A is linearly independent.

Problem 18. (5 points) Let $v_1, \ldots, v_p$ be a set of vectors in $\mathbb{R}^n$ and suppose A is an $m \times n$ matrix. Prove that if $Av_1, \ldots, Av_p$ is linearly independent, then $v_1, \ldots, v_p$ is linearly independent.

Problem 19. (5 points) Let M_n be the vector space of $n \times n$-matrices. The trace of a matrix $A \in M_n$ is the sum of the diagonal entries of A. Let $\mathrm{T} : M_n \to \mathbb{R}$ be function that takes a matrix to its trace. Prove that T is a linear transformation.

Problem 20. (5 points) Suppose $T : V \to V$ is a linear transformation. Prove that $\ker(T) \neq \{\mathbf{0}\}$ if and only if 0 is an eigenvalue of T.

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