University of South Carolina
MATH 544-001

Final Examination - Sample A

December 12, 2024

Closed book examination Time: 150 minutes

Instructions:

Notes, books, computer, phones, calculators or other aids are **not** allowed. Please write on only one side of each page. If you need more space than is provided, then ask for extra paper from the proctor. Simplify your final answers. Full credit will not be awarded for insufficient accompanying work.

There are 60 total points available.

Problem 1. (5 points) Find the reduced row echelon form for the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 4 & 2 \\ 2 & 0 & 1 & 8 & 0 \\ 1 & 2 & 1 & 5 & 6 \end{pmatrix}.$$

Problem 2. (5 points) Find a basis for the null space of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 1 \\ 2 & 4 & 6 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 2 & 2 \\ 1 & 4 & 3 & 0 & 1 & 1 \end{pmatrix}$$

Problem 3. (5 points) Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

For each of the following, compute the matrix or indicate that the expression is undefined.

AB

BA

$CB + CA$

$C^{-1} + C^T$

Problem 4. (5 points) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

or show that it does not exist.

Problem 5. (5 points) Find the determinant of the following matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 3 & 8 & 2 \\ 6 & 5 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Problem 6. (5 points) The set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ where

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 4 \end{pmatrix}.$$

Problem 7. (5 points) If it exists, find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$ where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 8. (1 point) A function $f : X \to Y$ is injective if and only if for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.

Problem 9. (1 point) A system of linear equations always has either no solutions, exactly one solution, or infinitely many solutions.

Problem 10. (1 point) A square matrix is invertible if and only if its columns are linearly independent.

Problem 11. (1 point) Let A be an $n \times m$ matrix. If there exists a matrix B such that $AB = I_n$, then $BA = I_m$ also.

Problem 12. (1 point) Let S be a finite set of vectors in $\mathbb{R}^n$. If S is linearly independent, then every non-empty subset of S is linearly independent.

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 13. (1 point) A set of vectors $\mathcal{B}$ in a vector space V is a basis for V if and only if $\mathcal{B}$ is linearly independent and is a spanning set for V.

Problem 14. (1 point) Let A and B be $n \times n$ matrices. Then $\det(A + B) = \det(A) + \det(B)$.

Problem 15. (1 point) If A is a square matrix, then $\mathrm{Nul}(A) = \mathrm{Col}(A)$.

Problem 16. (1 point) A square matrix A is invertible if and only if it is diagonalizable.

Problem 17. (1 point) Let A be a square matrix. If A has an eigenvalue λ with algebraic multiplicity 1, then the corresponding eigenspace E_λ has dimension 1.

Problem 18. (5 points) Let M_n be the vector space of $n \times n$ matrices. A matrix A in M_n is skew-symmetric if $A^T = -A$. Prove that the subset of skew-symmetric matrices are a subspace of M_n.

Problem 19. (5 points) Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove that if T is surjective, but not injective, then, for every $w \in W$, the equation $T(v) = w$ has infinitely many solutions $v \in V$.

Problem 20. (5 points) Prove that if λ is an eigenvalue of a square matrix A, then λ^n is an eigenvalue of A^n for every positive integer n.

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