Problem A. Let $T : V \to W$ be an invertible linear transformation from a vector space V to a vector space W. Prove that if $v_1, \dots, v_p$ is a basis for V, then $T(v_1), \dots, T(v_p)$ is a basis for W.

Solution. From Assignment 7B, we conclude that $T(v_1), \dots, T(v_p)$ is linearly independent. From Assignment 7C, we conclude that $T(v_1), \dots, T(v_p)$ spans W. Thus, it is a basis as desired.

Problem B. Let $\mathcal{B}$ be a finite basis for a vector space V. Prove that the vectors $u_1, \dots, u_p$ in V are linearly independent if and only if the coordinate vectors $[u_1]_{\mathcal{B}}, \dots, [u_p]_{\mathcal{B}}$ are linearly independent.

Solution. Recall that the map $T : V \to \mathbb{R}^n$ given by $T(x) = [x]_{\mathcal{B}}$ is an invertible linear transformation. Thus, this follows immediately from Problem A.

Problem C. Let U, V, W be vector spaces and let $T : U \to V$ and $S : V \to W$ be linear transformations. Prove that if $\text{im}(T) \cap \ker(S) = \{0\}$ then $\ker(T) = \ker(S \circ T)$.

Solution. Suppose $x \in \ker(T)$. T hen $S(T(x)) = S(0) = 0$. Thus $x \in \ker(S \circ T)$.

Now suppose $x \in \ker(S \circ T)$. Then $S(T(x)) = 0$. Then $T(x) \in \ker(S)$. Since also $T(x) \in \text{im}(T)$, we conclude that $T(x) \in \{0\}$. Thus $T(x) = 0$. Thus $x \in \ker(T)$.