Problem A. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Prove that the range of T is a subspace of W.

Solution. Let U be the range of T. We check the properties given in the definition of being a subspace. First, we observe that $T(0) = 0$ so $0 \in W$. Second, if $u, v \in U$, then there exist $x, y \in V$ such that $u = T(x)$ and $v = T(y)$. Thus $u + v = T(x) + T(y) = T(x + y)$ is in the range of T. Finally, if $c \in \mathbb{R}$, then $cu = cT(x) = T(cx)$ is in the range of T. We conclude that U is a subspace.

Problem B. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Suppose $v_1, \ldots, v_p$ is a linearly independent set in V. Prove that, if T is injective, then $T(v_1), \ldots, T(v_p)$ is linearly independent.

Solution. We prove that contrapositive. Suppose $T(v_1), \ldots, T(v_p)$ is linearly dependent. Then there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1T(v_1) + \cdots + c_pT(v_p) = 0 \ .$$

By linearity, this means

$$T(c_1v_1 + \cdots + c_pv_p) = 0 \ .$$

Since T is injective, this means

$$c_1v_1 + \cdots + c_pv_p = 0 \ .$$

This means $v_1, \ldots, v_p$ is linearly dependent.

Problem C. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Suppose $v_1, \ldots, v_p$ spans V. Prove that, if T is surjective, then $T(v_1), \ldots, T(v_p)$ spans W.

Solution. Consider some $w \in W$. Since T is surjective, there exists a $v \in V$ such that $w = T(v)$. Since $v_1, \ldots, v_p$ span V, there exist weights $c_1, \ldots, c_p$ such that

$$v = \sum_{i=1}^{p} c_iv_i \ .$$

Thus

$$w = T(v) = T(\sum_{i=1}^{p} c_iv_i) = \sum_{i=1}^{p} c_iT(v_i).$$

We conclude that w is a linear combination of $T(v_1), \ldots, T(v_p)$. Since w was arbitrary, they form a spanning set for W as desired.