Problem A. A matrix A is symmetric if $A = A^T$. Prove that the set of symmetric matrices form a subspace of the vector space of all $n \times n$ matrices.

Solution. Let V be the vector space of all $n \times n$ matrices and let S be the subset of symmetric matrices. We verify the three properties in the definition of a subspace. First, $0^T = 0$ so the zero matrix is symmetric. Second, if A, B are symmetric matrices, then $(A+B)^T = A^T + B^T = A+B$; thus $A+B$ is also symmetric. Finally, if c is a scalar and A is a symmetric matrix, then $(cA)^T = cA^T = cA$; thus cA is symmetric.

Problem B. Let U, V be subspaces of a vector space W. Prove that $U \cap V$ is a subspace of W. Find an example where $U \cup V$ is not a subspace of W.

Solution. We verify the three defining properties of a vector space. Since $0 \in U$ and $0 \in V$, we conclude $0 \in U \cap V$. If $u, v \in U \cap V$, then $u, v \in U$ and $u, v \in V$. Since U is a vector space, $u + v \in U$. Since V is a vector space, $u + v \in V$. Thus $u + v \in U \cap V$. If $u \in U \cap V$ and $c \in \mathbb{R}$, then $cu \in U$ and $cu \in V$. Thus $cu \in U \cap V$.

For the second statement, observe that

$$U = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}, \quad W = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \;\middle|\; y \in \mathbb{R} \right\}$$

are subspaces of $\mathbb{R}^2$. Consider:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u + v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^2$. Observe that $u \in U \subseteq U \cup V$ and $v \in V \subseteq U \cup V$, but $u + v$ is in neither U nor V. Thus $U \cup V$ is not a subspace.