Problem A. Let X and Y be nonempty sets. Let $f : X \to Y$ be an injective function. Prove that there exists a function $g : Y \to X$ such that $g \circ f$ is the identity on X. Give an explicit example to show that g is not necessarily unique.

Solution.
Since X is nonempty, we may choose an element $z \in X$. We define $g : Y \to X$ as follows

$$g(y) = \begin{cases} x & \text{if there exists } x \in X \text{ such that } f(x) = y \\ z & \text{otherwise.} \end{cases}$$

The above definition only makes sense because g is injective. Recall that a function must have a unique output for each input. When y is not in the range of f, we have the unique output z. When y is in the range of f, then there is at most one x such that $f(x) = y$; thus there is a unique output. For any $x \in X$, we see that $g(f(x)) = x$ as desired.

To see that g is not necessarily unique, consider $X = \{1, 2\}$ and $Y = \{3, 4, 5\}$ with $f(1) = 3$ and $f(2) = 4$. We must define $g(3) = 1$ and $g(4) = 2$, but $g(5)$ can be either 1 or 2 and still satisfy the desired property.

Problem B. Let X and Y be nonempty sets. Let $f : X \to Y$ be a surjective function. Prove that there exists a function $h : Y \to X$ such that $f \circ h$ is the identity on Y. Give an explicit example to show that h is not necessarily unique.

Solution.
Since f is surjective, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Define a function $h : Y \to X$, where for each $y \in Y$ we define $h(y) = x$ for some choice $x \in X$ such that $f(x) = y$. Observe that $f(h(y)) = y$ for every $y \in Y$. (Technically speaking, this is an application of the Axiom of Choice, but this is beyond the scope of the course.)

To see that h is not necessarily unique, consider $X = \{1, 2\}$ and $Y = \{3\}$ with $f(1) = 3$ and $f(2) = 3$. We see that $h(3)$ can be either 1 or 2 and still satisfy the desired property.