Problem A. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that $(AB)^T = B^T A^T$.

Solution.

By the definition of transpose, we have $(C^T)_{ij} = C_{ji}$ for any matrix C with appropriate indices i, j. Thus, for every $1 \leq i \leq p$ and $1 \leq j \leq m$, we have

$$\left[(AB)^T\right]_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} = \left(B^T A^T\right)_{ij}.$$

Since all entries agree, the two matrices agree.

Problem B. Prove that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

Solution.

First, suppose $ad - bc \neq 0$ and let $B = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We check that $AB = I_2$ and $BA = I_2$, so B is the inverse of A. Thus A is invertible.

Now suppose $ad - bc = 0$. Observe that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ ad - bc \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution unless $a = b = 0$. If $a = b = 0$, then

$$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \begin{pmatrix} -d \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and again there is a non-trivial solution unless $c = d = 0$. If $a = b = c = d = 0$, then $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$. In all these cases, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Thus A is not invertible by (the contrapositive of) Theorem 2.5 in the text.