Problem A. Let m and b be real numbers. Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be the function given by $f(x) = mx + b$. Prove that f is a linear transformation if and only if $b = 0$.

Solution.

Suppose f is a linear transformation. Then $f(2) = 2f(1)$. Since $f(2) = 2m + b$ and $f(1) = m + b$, we have $2m + b = 2(m + b)$, which simplifies to $b = 0$ as desired.

Now suppose that $b = 0$. For any $u, v \in \mathbb{R}^1$, we have

$$f(u + v) = m(u + v) = mu + mv = f(u) = f(v).$$

For any $u \in \mathbb{R}^1$ and $c \in \mathbb{R}$, we have

$$f(cu) = m(cu) = c(mu) = cf(u).$$

Thus, by the definition, f is a linear transformation.

Problem B. Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that if $g \circ f$ is injective, then f is injective.

Solution.

Suppose $g \circ f$ is injective. This means that for every $z \in Z$, there is at most one $x \in X$ such that $(g \circ f)(x) = z$. Consider $y \in Y$. Suppose $x, x' \in X$ are elements such that $f(x) = y = f(x')$. Applying g, we see that $g(f(x)) = g(f(x'))$. Since $(g \circ f)$ is injective, this means that $x = x'$. Therefore there is at most one $x \in X$ such that $f(x) = y$. Therefore, f is injective.