Problem A. Let a, b be real numbers and let $\mathbf{u}, \mathbf{v}$ be column vectors in $\mathbb{R}^n$. Prove that $(a+b)(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + b\mathbf{u} + a\mathbf{v} + b\mathbf{v}$.

Solution.
We have

$$(a+b)(\mathbf{u}+\mathbf{v}) = (a+b)\mathbf{u} + (a+b)\mathbf{v}$$

by Property (v) from page 53 of the text. Then, we have

$$(a+b)\mathbf{u} + (a+b)\mathbf{v} = a\mathbf{u} + b\mathbf{u} + a\mathbf{v} + b\mathbf{v}$$

by two applications of property (vi) from page 53 of the text.

Problem B. Prove that $\operatorname{Span}(X \cap Y) \subseteq \operatorname{Span}(X) \cap \operatorname{Span}(Y)$.

Solution.
Suppose $\mathbf{v}$ is an element of $\operatorname{Span}(X \cap Y)$. By definition, this means that $\mathbf{v}$ is a linear combination of elements of $X \cap Y$. This means that

$$\mathbf{v} = \sum_{\mathbf{w} \in X \cap Y} c_{\mathbf{w}} \mathbf{w}$$

where each $c_{\mathbf{w}}$ is a real number. Observe that

$$\mathbf{v} = \sum_{\mathbf{w} \in X \cap Y} c_{\mathbf{w}} \mathbf{w} + \sum_{\mathbf{w} \in X \setminus Y} 0\mathbf{w},$$

so $\mathbf{v}$ is a linear combination of the elements in X. Thus $\mathbf{v} \in \operatorname{Span}(X)$. Similarly,

$$\mathbf{v} = \sum_{\mathbf{w} \in X \cap Y} c_{\mathbf{w}} \mathbf{w} + \sum_{\mathbf{w} \in Y \setminus X} 0\mathbf{w},$$

so $\mathbf{v} \in \operatorname{Span}(Y)$. We conclude that $\mathbf{v} \in \operatorname{Span}(X) \cap \operatorname{Span}(Y)$.