Due: Thursday, April 19, 2018

Read §4.1 – 4.5 from the text.

Problem 1 (c.f. 4.1.2) Let $J = \langle x^2 + y^2 - 1,\ y - 1 \rangle$. Find $f \in \mathbf{I}(\mathbf{V}(J))$ such that $f \notin J$.

Problem 2 (c.f. 4.2.6) Let I be an ideal in $k[x_1, \ldots, x_n]$.

(a) In the special case when $\sqrt{I} = \langle f_1, f_2 \rangle$, with $f_i^{m_i} \in I$, prove that $f^{m_1+m_2-1} \in I$ for all $f \in \sqrt{I}$.

(b) Now prove that for any I, there exists a single integer m such that $f^m \in I$ for all $f \in \sqrt{I}$.

Problem 3 (c.f. 4.2.7) Determine whether the following polynomials lie in the following radicals. If the answer is yes, what is the smallest power of the polynomial that lies in the ideal?

(a) Is $x + y \in \sqrt{\langle x^3,\ y^3,\ xy(x+y) \rangle}$?

(b) Is $x^2 + 3xy \in \sqrt{\langle x + z,\ x^2 y,\ x - z^2 \rangle}$?

Problem 4 (c.f. 4.3.7) Let I and J be ideals in $k[x_1, \ldots, x_n]$. Prove the following:

(a) If $I^l \subseteq J$ for some integer $l > 0$, then $\sqrt{I} \subseteq \sqrt{J}$.

(b) $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

Problem 5 (c.f. 4.3.8) Let

$$f = x^4 + x^3 y + x^3 z^2 - x^2 y^2 + x^2 y z^2 - x y^3 - x y^2 z^2 - y^3 z^2$$

and

$$g = x^4 + 2x^3 z^2 - x^2 y^2 + x^2 z^4 - 2 x y^2 z^2 - y^2 z^4\ .$$

(a) Compute generators for $\langle f \rangle \cap \langle g \rangle$.

(b) Compute generators for $\sqrt{\langle f \rangle \langle g \rangle}$.

(c) Compute $\gcd(f, g)$.

(d) Calculate $\langle f, g \rangle \cap \langle x^2 + xy + xz + yz,\ x^2 - xy - xz + yz \rangle$.

Problem 6 (c.f. 4.3.9) Show that $\sqrt{IJ} = \sqrt{I \cap J}$. Given an example to show that the product of radical ideals need not be radical. Also give an example to show that $\sqrt{IJ}$ can differ from $\sqrt{I}\sqrt{J}$.

Problem 7 (c.f. 4.4.4) Let I and J be ideals in $k[x_1, \ldots, x_n]$. Show that if I is radical, then $I : J$ is radical and $I : J = I : \sqrt{J} = I : J^\infty$.

Problem 8 (c.f. 4.5.3) Show that an ideal I is prime if and only if for any ideals J and K such that $JK \subseteq I$, either $J \subseteq I$ or $K \subseteq I$.

Bonus Problem 1 (c.f. 4.3.11) Two ideals I and J of $k[x_1, \ldots, x_n]$ are said to be *comaximal* if and only if $I + J = k[x_1, \ldots, x_n]$.

(a) Show that if $k = \mathbb{C}$, then I and J are comaximal if and only if $\mathbf{V}(I) = \mathbf{V}(J)$. Give an example to show that this is false in general.

(b) Show that if I and J are comaximal, then $IJ = I \cap J$.

(c) If $IJ = I \cap J$, does it necessarily follow that I and J are comaximal?

(d) Show that I^r and J^s are comaximal for all positive integers r and s.

(e) Let $I_1, \ldots, I_r$ be ideals in $k[x_1, \ldots, x_n]$ and suppose that I_i and $J_i = \cap_{j \neq i} I_j$ are comaximal for all i. Show that

$$I_1^m \cap \cdots \cap I_r^m = (I_1 \cdots I_r)^m = (I_1 \cap \cdots \cap I_r)^m$$

for all positive integers m.