Due: Thursday, March 29, 2018

Read §3.1 – 3.5 from the text.

Problem 1 (c.f. 3.1.2) Consider the system of equations

$$x^2 + 2y^2 = 3$$
$$x^2 + xy + y^2 = 3\ .$$

(a) If I is the ideal generated by these equations, then find $I \cap k[x]$ and $I \cap k[y]$.
(b) Find all complex solutions of these equations.
(c) Find all rational solutions of these equations.

Problem 2 (c.f. 3.1.4) Find bases for the elimination ideals I_1 and I_2 for the ideal I determined by the equations

$$x^2 + y^2 + z^2 = 4$$
$$x^2 + 2y^2 = 5$$
$$xz = 1\ .$$

How many rational solutions are there?

Problem 3 (c.f. 3.1.9) Consider the system of equations given by

$$x^5 + \frac{1}{x^5} = y$$
$$x + \frac{1}{x} = z\ .$$

Let I be the ideal in $\mathbb{C}[x, y, z]$ determined by these equations.

(a) Find a basis of $I_1 \subseteq \mathbb{C}[y, z]$ and show that $I_2 = \{0\}$.
(b) Use the extension to prove that each partial solution $c \in \mathbf{V}(I_2) = \mathbb{C}$ extends to a solution in $\mathbf{V}(I) \subseteq \mathbb{C}^3$.
(c) Which partial solutions $(b, c) \in \mathbf{V}(I_1) \subseteq \mathbb{R}^2$ extend to solutions in $\mathbf{V}(I) \subseteq \mathbb{R}^3$?

Problem 4 (c.f. 3.2.4) To see how the Closure Theorem can fail over $\mathbb{R}$, consider the ideal

$$I = \langle x^2 + y^2 + z^2 + 2, 3x^2 + 4y^2 + 4z^2 + 5\rangle\ .$$

Let $V = \mathbf{V}(I)$, and let π_1 be the projection taking (x, y, z) to (y, z).

(a) Working over $\mathbb{C}$, prove that $\mathbf{V}(I_1) = \pi_1(V)$.
(b) Working over $\mathbb{R}$, prove that $V = \emptyset$ and that $\mathbf{V}(I_1)$ is infinite.

Problem 5 (c.f. 3.3.7) Let S be the parametric surface

$$x = uv, \quad y = uv^2, \quad z = u^2 \ .$$

(a) Find the equation of the smallest variety V that contains S.
(b) Over $\mathbb{C}$, determine exactly which points of V are not on S.

Problem 6 (c.f. 3.3.9) The *Whitney umbrella surface* S is given parametrically by

$$x = uv, \quad y = v, \quad z = u^2 \ .$$

(a) Find the equation of the smallest variety V containing the Whitney umbrella.
(b) Show that $V = S$ over the $\mathbb{C}$, but not over $\mathbb{R}$.
(c) Show that the parameters u and v are not always uniquely determined by x, y, z.

Problem 7 (c.f. 3.3.11) Prove Theorem 2 from §3.3 in the text. (Hint: the proof is outlined in the text and suggestions for filling in the details can be found in Exercise 11 from §3.3.)

Problem 8 (c.f. 3.4.8)
(a) Show that $(0, 0)$ is the *only* singular point of $y^2 = x^3$.
(b) Find all singular points of the curve $y^2 = cx^2 - x^3$ for all values of c.
(c) Find all singular points of the curve $x^2 + y^2 = c$ for all values of c.

Bonus Problem 1 Recall that a *Pythagorean triple* is a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. A triple is *primitive* if $\gcd(a, b, c) = 1$.
(a) Find a bijection between primitive Pythagorean triples and rational solutions to the equation $x^2 + y^2 = 1$ where $x > 0$ and $y > 0$.
(b) Show the rational parametrization

$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}$$

is a curve whose Zariski closure is the unit circle $C = \mathbf{V}(x^2 + y^2 - 1)$.
(c) Show that there is exactly one point p in $\mathbf{V}(x^2 + y^2 - 1)$ that is not contained in the parametrization.
(d) Show that every rational point on $C \setminus \{p\}$ is the image of exactly one value $t \in \mathbb{Q}$.
(e) Find a bijection between primitive Pythagorean triples and the set of rational numbers $0 < r < 1$.

Bonus Problem 2 (c.f. 3.4.12) Consider a surface $\mathbf{V}(f) \subset k^3$ defined by $f \in k[x, y, z]$.
(a) Define what it means for $(a, b, c) \in \mathbf{V}(f)$ to be a singular point.
(b) Determine all singular points of the sphere $x^2 + y^2 + z^2 = 1$.
(c) Determine all singular points on the surface $\mathbf{V}(x^2 - y^2 z + z^3)$.