Due: Thursday, March 8, 2018

Read §2.5–2.10 from the text.

You will definitely need to use computer algebra for Problems 4–8! Don't forget to cite any resources you use. Also, recall that you can hand in annotated print-outs from the computer algebra system.

Problem 1 (c.f. 2.6.1) Fix a monomial ordering an let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Suppose that $f \in k[x_1, \ldots, x_n]$. Show that there is a *unique* expression $f = g + r$ where $g \in I$ and no term of r is divisible by any element of $\mathrm{LT}(I)$. Thus, once a monomial order is fixed, there is a unique "remainder of f on division by I".

Problem 2 (c.f. 2.6.5) Compute $S(f, g)$ using the lex order.

(a) $f = 4x^2z - 7y^2$, $g = xyz^2 + 3xz^4$.
(b) $f = x^4y - z^2$, $g = 3xz^2 - y$.
(c) $f = x^7y^2z + 2xyz$, $g = 2x^7y^2z + 4$.

Problem 3 (c.f. 2.6.9) Show that $\{y - z^2, z - x^3\}$ is not a Gröbner basis for lex order.

Problem 4 (c.f. 2.7.2–3) Find reduced Gröbner bases for the following ideal using both the lex and grlex monomial orderings.

(a) $\langle x^2y - 1, xy^2 - x\rangle$.
(b) $\langle x^2 + y, x^4 + 2x^2y + y^2 + 3\rangle$.
(c) $\langle x - z^4, y - z^5\rangle$.

Problem 5 (c.f. 2.8.1) Determine whether $f = xy^3 - z^2 + y^5 - z^3$ is in the ideal $I = \langle -x^3 + y,\ x^2y - z\rangle$.

Problem 6 (c.f. 2.8.3) Find the points in $\mathbb{C}^2$ on the variety

$$\mathbf{V}(x^2 + y^2 + z^2 - 1, x^2 + y^2 + z^2 - 2x, 2x - 3y - z)\ .$$

Problem 7 (c.f. 2.8.8) Find a proper affine variety in $\mathbb{R}^3$ containing the parametric surface

$$\begin{aligned} x &= (2 + \cos(t))\cos(u) \\ y &= (2 + \cos(t))\sin(u) \\ z &= \sin(t) \end{aligned}$$

where t and u are real parameters.

Problem 8 (c.f. 2.8.11) Suppose a, b, c are real numbers such that

$$
\begin{aligned}
a + b + c &= 3 \\
a^2 + b^2 + c^2 &= 5 \\
a^3 + b^3 + c^3 &= 7 \ .
\end{aligned}
$$

Find $a^4 + b^4 + c^4$, $a^5 + b^5 + c^5$, and $a^6 + b^6 + c^6$.

Bonus Problem 1 (c.f. 2.6.13) Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, and let G be a Gröbner basis of I.

(a) Show that $\overline{f}^G = \overline{g}^G$ if and only if $f - g \in I$.

(b) Show that $\overline{f+g}^G = \overline{f}^G + \overline{g}^G$.

(c) Show that $\overline{fg}^G = \overline{\overline{f}^G \overline{g}^G}$.

This shows how to do calculations in the *quotient ring* $k[x_1, \ldots, x_n]/I$, which we may see later in the course.

Bonus Problem 2 (c.f. 2.8.9) Find a 1-dimensional affine variety in $\mathbb{R}^3$ containing the parametric curve

$$
\begin{aligned}
x &= (2 + \cos(2s)) \cos(3s) \\
y &= (2 + \cos(2s)) \sin(3s) \\
z &= \sin(2s) \ .
\end{aligned}
$$

(The final answer is surprisingly complicated!)

Bonus Problem 3 A set $G = \{g_1, \ldots, g_s\}$ of polynomials in $k[x_1, \ldots, x_n]$ is called a *universal Gröbner basis* if it is a Gröbner basis for every monomial order simultaneously.

(a) Find a set G that is a Gröbner basis in *lex* order, but not in *grlex* order.

(b) Find a set G that is a Gröbner basis in *grlex* order, but not in *lex* order.

(c) Prove that every ideal has a universal Gröbner basis. (Hint: Dickson's Lemma.)