Due: Thursday, February 15, 2018

Read §2.1–2.4 from the text.

You do not need to use a computer algebra system to do these problems. Make sure your solutions can be checked by hand.

Problem 1 (c.f. 2.1.1) For each of the following, determine whether $f \in \mathbb{R}[x]$ is contained in the ideal $I \subseteq \mathbb{R}[x]$.

(a) $f = x^2 - 3x + 2,\ I = \langle x - 2\rangle$.
(b) $f = x^5 - 4x + 1,\ I = \langle x^3 - x^2 + x\rangle$.
(c) $f = x^2 - 4x + 4,\ I = \langle x^4 - 6x^2 + 12x - 8, 2x^3 - 10x^2 + 16x - 8\rangle$.
(d) $f = x^3 - 1,\ I = \langle x^9 - 1, x^5 + x^3 - x^2 - 1\rangle$.

Problem 2 (c.f. 2.2.1, 2.2.3) Rewrite each of the following polynomials in lex, grlex, and grevlex order. Then repeat with the variables ordered $z > y > x$ instead. (There are 6 rewrites for each polynomial.)

(a) $f(x, y, z) = 2x + 3y + z + x^2 - z^2 + x^3$.
(b) $f(x, y, z) = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$.

Problem 3 (c.f. 2.2.11) Let $>$ be a monomial order on $k[x_1, \ldots, x_n]$.

(a) Suppose $f \in k[x_1, \ldots, x_n]$ is nonzero and m is a nonzero monomial. Show that $\mathrm{LT}(m \cdot f) = m \cdot \mathrm{LT}(f)$.
(b) Suppose $f, g \in k[x_1, \ldots, x_n]$ are nonzero. Is it true that $\mathrm{LT}(fg) = \mathrm{LT}(f)\,\mathrm{LT}(g)$? If so, prove it; otherwise, find a counterexample.
(c) Suppose $f_1, \ldots, f_s, g_1, \ldots g_s \in k[x_1, \ldots, x_n]$ and $\sum_{i=1}^{s} f_i g_i$ are all non-zero. Is it true that $\mathrm{LT}(\sum_{i=1}^{s} f_i g_i)$ is equal to $\mathrm{LT}(f_i)\,\mathrm{LT}(g_i)$ for some i? If so, prove it; otherwise, find a counterexample.

Problem 4 (c.f. 2.2.12) Prove Lemma 2.2.8 from the text.

Problem 5 (c.f. 2.3.1) Compute the remainder on division of $f = x^3y^2 + xy^3 - y + 1$ by the ordered set $F = (xy^2 - x, x - y^3)$ using both the lex and grlex monomial orders.

Problem 6 (c.f. 2.3.2) Compute the remainder on division of $f = xy - 2yz$ by the ordered set $(x - y^2, y - z^3, z^2 - 1)$ using to the lex monomial order.

Problem 7 (c.f. 2.4.10) Let $\mathbf{u} = (u_1, \ldots, u_n)$ be a vector in $\mathbb{R}^n$ such that $u_1, \ldots, u_n$ are positive and linearly independent over $\mathbb{Q}$. For $\alpha, \beta \in \mathbb{N}^n$, define

$$\alpha >_{\mathbf{u}} \beta \text{ iff } \mathbf{u} \cdot \alpha > \mathbf{u} \cdot \beta$$

where the dot is the usual dot product of vectors. Prove that $>_{\mathbf{u}}$ is a monomial order.

Problem 8 (c.f. 2.4.11) Let $\mathbf{u} = (u_1, \ldots, u_n)$ be in $\mathbb{N}^n$, and fix a monomial order $>_\sigma$ on $\mathbb{N}^n$. For $\alpha, \beta \in \mathbb{N}^n$, define $\alpha >_{\mathbf{u},\sigma} \beta$ if and only if $\mathbf{u} \cdot \alpha > \mathbf{u} \cdot \beta$ or $\mathbf{u} \cdot \alpha = \mathbf{u} \cdot \beta$ and $\alpha >_\sigma \beta$.

(a) Prove that $>_{\mathbf{u},\sigma}$ is a monomial order.
(b) Find $\mathbf{u} \in \mathbb{N}^n$ so that $>_{\mathbf{u},lex}$ is the grlex order $>_{grlex}$.
(c) Note that $>_\sigma$ is used break ties. Prove that, if $n \geq 2$, then for all $\mathbf{u} \in \mathbb{N}^n$ there exist $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha \neq \beta$ but $\mathbf{u} \cdot \alpha = \mathbf{u} \cdot \beta$.

Problem 9 (Bonus) The previous two problems give examples of *weight orders*, which we now define in general. Given a sequence S of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}^n$, define an order $>_S \beta$ as follows. If $\alpha \cdot \mathbf{u}_1 > \beta \cdot \mathbf{u}_1$, then $\alpha >_S \beta$; if $\alpha \cdot \mathbf{u}_1 < \beta \cdot \mathbf{u}_1$, then $\alpha <_S \beta$; otherwise, we must have $\alpha \cdot \mathbf{u}_1 = \beta \cdot \mathbf{u}_1$. In that case, we now compare $\alpha \cdot \mathbf{u}_2$ and $\beta \cdot \mathbf{u}_2$ similarly. If it is still a tie, we use $\mathbf{u}_3$ and so on.

For certain sequences S, the resulting relation $>_S$ is a monomial order. In fact, *every* monomial order can be described in this way (we won't prove this).

(a) Show that lex, grlex, and grevlex are weight orders by explicitly describing appropriate sequences S of vectors in $\mathbb{R}^n$.
(b) Find examples of sequences S such that $>_S$ is *not* a monomial order.