Due: Thursday, February 1, 2018

Read §1.1–1.5 from the text. You may need to use ideas from the text not directly discussed in lecture to complete the problem set. In particular, there may be hints in the corresponding problem in the textbook! Note that some questions may be slightly different than the corresponding question in the textbook.

Reminder: You may use any resources you want to complete the assignment, however you must:

- write up the problem set alone in your own words, and
- list any resources used to do the assignment.

If you use a computer algebra system to complete a problem, be sure to include both the input and output.

Problem 1 (c.f. Exercise 1.1.5) Consider the polynomial

$$f(x,y,z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3 .$$

(a) Write f as a polynomial in x with coefficients in $k[y,z]$.
(b) Write f as a polynomial in y with coefficients in $k[x,z]$.
(c) Write f as a polynomial in z with coefficients in $k[x,y]$.

Problem 2 (c.f. Exercise 1.2.7) Consider the curve C defined by the polar equation $r = \sin(2\theta)$. Prove that C is the affine variety $\mathbf{V}((x^2+y^2)^3 - 4x^2y^2)$.

Problem 3 (c.f. Exercise 1.2.8) Prove that the set

$$X = \{(x,x) \mid x \in \mathbb{R}, x \neq 1\} \subseteq \mathbb{R}^2$$

is *not* an affine variety.

Problem 4 (c.f. Exercise 1.3.4) Consider the parametric representation

$$x = \frac{t}{1+t}, \quad y = 1 - \frac{1}{t^2} .$$

Find the equation of the affine variety determined by the above parametric equations.

Problem 5 (c.f. Exercise 1.4.1) Consider the equations

$$x^2 + y^2 - 1 = 0, \quad xy - 1 = 0 .$$

(a) Use algebra to eliminate y from the above equations.
(b) Show how the polynomial from part (a) lies in $\langle x^2+y^2-1, xy-1\rangle$.

Problem 6 (c.f. Exercise 1.4.3) *Without* using a computer algebra system, prove the following equalities in $\mathbb{Q}[x, y]$:

(a) $\langle x + y, x - y\rangle = \langle x, y\rangle$.
(b) $\langle x + xy, y + xy, x^2, y^2\rangle = \langle x, y\rangle$.
(c) $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3\rangle = \langle x^2 - 4, y^2 - 1\rangle$.

(You may use the result of Exercise 1.4.2 without proving it.)

Problem 7 (c.f. Exercise 1.5.8) Using a computer algebra system, compute the following gcd's:

(a) $\gcd(x^4 + x^2 + 1, x^4 - x^2 - 2x - 1, x^3 - 1)$.
(b) $\gcd(x^3 + 2x^2 - x - 2, x^3 - 2x^2 - x + 2, x^3 - x^2 - 4x + 4)$.

Problem 8 (c.f. Exercise 1.5.9) *Without* using a computer algebra system, determine whether $x^2 - 4$ is contained in the ideal

$$\langle x^3 + x^2 - 4x - 4,\ x^3 - x^2 - 4x + 4,\ x^3 - 2x^2 - x + 2\rangle\ .$$

Problem 9 (Bonus) Let X be the *Fermat cubic surface*

$$\mathbf{V}(x^3 + y^3 + z^3 + 1) \subset \mathbb{C}^3$$

and let Y be the *Clebsch diagonal cubic surface*

$$\mathbf{V}(x^3 + y^3 + z^3 + 1 - (x + y + z + 1)^3) \subset \mathbb{C}^3$$

Let L be the *line* $\mathbf{V}(x + y, z + 1)$. (Note that L is a "line" since it is defined by linear equations and has complex dimension 1.)

(a) Show that X and Y both contain L.
(b) Find 27 lines in X. (Hints: First, look for planes in the three-dimensional variety $\mathbf{V}(x^3 + y^3 + z^3 + w^3) \subset \mathbb{C}^4$. How does $x^3 + y^3$ factor in $\mathbb{C}[x, y]$? Take advantage of symmetry!)

There are actually 24 lines in Y and they can all be defined over the real numbers.