The midterm will be on material from §2.6–2.8, 3.1–3.4 from the text. You do not need to know anything about envelopes. The questions below are intended to give you a rough idea of what you can expect on the midterm. The actual midterm will not be of the same length, nor will it have exactly the same types of questions.

The emphasis will be on computational skills, but some questions will demand a deeper understanding of the material. Note that **you will not be able to use computer algebra during the midterm**, so make sure you can do all of these problems by hand!

**Problem 1** Find the $S$-polynomial $S(f, g)$ for each of the following pairs of polynomials $f, g \in \mathbb{Q}[x, y, z]$ for the *lex* order.

(a) $f = x^2y - 2xy$, $g = xz^2 - 3y^2$
(b) $f = 4xy - 3y^2z$, $g = 3yz - z^3$

**Problem 2** Determine whether each of the following sets of polynomials are a Gröbner basis in $\mathbb{Q}[x, y, z]$ for the *lex* order.

(a) $\{x - 3y^2,\ y^3 - 4y\}$
(b) $\{xy - 2y^3z,\ y^3 - z\}$
(c) $\{x^2 - y,\ y^2 - yz,\ xy - 3\}$

**Problem 3** For each of the following ideals in $\mathbb{Q}[x, y, z]$, find a Gröbner basis in the *lex* order.

(a) $\langle x + y,\ x - y + z,\ y - z\rangle$
(b) $\langle x - y^3,\ -x^2 + xy^2\rangle$
(c) $\langle x^2 - xy,\ xy - 1\rangle$

**Problem 4** The following sets are Gröbner bases in $\mathbb{Q}[x, y, z]$ for the *lex* order. For each set, find the *reduced* Gröbner basis generating the same ideal.

(a) $\{x^2 - xy,\ y^2 + y,\ xy + y^3,\ x + y,\ y\}$
(b) $\{3x + 4,\ 4y^3 - 2y + 2\}$
(c) $\{xy + y^3 - z - 2z^2,\ xz + y^3z - 2y^2z^2 - z^2,\ y^3 - z\}$

**Problem 5** The set

$$G = \{xy - z^2,\ xz - y^2,\ y^3 - z^3\}$$

is a Gröbner basis for an ideal $I \subseteq \mathbb{Q}[x, y, z]$ in *lex* order. Determine which of the following polynomials $f$ are contained in the ideal $I$.

(a) $f = xy - z^2$
(b) $f = x^2 + 3$
(c) $f = xy^3z - y^2z^3 - y^3z^3 + z^6$

**Problem 6** Suppose $I = \langle x^2 - y,\ x^2y - z \rangle \subseteq \mathbb{Q}[x, y, z]$. Find bases for the following ideals:

(a) $I \cap k[y, z]$
(b) $I \cap k[x, z]$
(c) $I \cap k[x, y]$
(d) $I \cap k[z]$

**Problem 7** Each of the following is a lex Gröbner basis for an ideal $I$ in $\mathbb{C}[x, y, z]$. In each case, describe the set of all partial solutions $(y, z) = (a_2, a_3)$ which extend to full solutions $(a_1, a_2, a_3) \in \mathbb{V}(I)$.

(a) $\{5z^2 - 7,\ 2y - 3z,\ 12x - 4z\}$
(b) $\{x^2 - z - 1,\ xy - z^2 - 3z - 2,\ xz + 2x - y,\ y^2 - z^3 - 5z^2 - 8z - 4\}$
(c) $\{xz - x + 3y,\ 3xy - z - 2,\ 9y^2 + z^2 + z - 2\}$

**Problem 8** Determine all singular points of $\mathbb{V}(6x^2 - xy - x - y^2 + 3y - 2) \subseteq \mathbb{R}^2$.