The midterm will be on material from §1.1–1.5, 2.1–2.5 from the text. The emphasis will be on computational skills, but some questions will demand a deeper understanding of the material. The questions below are intended to give you a rough idea of what you can expect on the midterm. The actual midterm will not be of the same length, nor will it have exactly the same types of questions.

Problem 1 Using the division algorithm, compute the quotient and remainder of division by f by g.
(a) $f = 2x^4 - 3x^2 - 3,\ g = x - 3.$
(b) $f = x^3 + 4x^2 - 2x + 1,\ g = 2x^2 - 2.$

Problem 2 Compute the following greatest common divisors.
(a) $\gcd(x^4 - 2x + 1, x^2 + 2x - 3).$
(b) $\gcd(x^3 - 3x - 2, x^3 - x^2 - x + 1).$
(c) $\gcd(x^6 - 1, x^4 - 1, x^3 - 1).$

Problem 3 Reorder each of the following polynomials using the *lex*, *grlex*, and *grevlex* monomial orders.
(a) $x^3 - 4x + x^4 - 3x^7.$
(b) $x^2y + 4xyz^2 + 2x^2yz^2 + x^3yz + 4.$

Problem 4 Determine whether the polynomial f is an element of the ideal I.
(a) $f = x^3 - 8,\ I = \langle x - 2\rangle.$
(b) $f = x^3 - 4x^2 + x + 6,\ I = \langle x^2 - 5x + 6\rangle.$
(c) $f = x^2,\ I = \langle x^3 - x, x^2 + 4x\rangle.$
(d) $f = x,\ I = \langle x^3 - x^2, x^4 + 4x^3\rangle.$
(e) $f = x^3 - 7x + 6,\ I = \langle x^2 - 5x + 6, x^2 - 4\rangle.$
(f) $f = x^3 - 3x^2 - x + 3,\ I = \langle x^2 - 5x + 6, x^2 - 4\rangle.$

Problem 5 Find a minimal basis of monomials for each of the following monomial ideals.
(a) $\langle x^6, x^4, x^{10}, x^{12}\rangle.$
(b) $\langle x^2y, x^3y^8, y^3z, xy^3z^2, y^2z^2, x^2yz\rangle.$
(c) $\langle 2y^3 - y, 2y^3 - x, x + y\rangle$ (Yes, this is a monomial ideal!)

Problem 6 Compute the remainder of f on division by the ordered set F using the division algorithm with the given monomial order.
(a) $f = 4x^3 - 2xy + y^2,\ F = (x^2 - y, xy - y^3)$ using *lex*.
(b) $f = 3x^4 - 2xy^2 + x^8,\ F = (x^3 - 3xy, y^2 - 2x)$ using *grlex*.

Problem 7 Find examples of each of the following. Justify your answers.

(a) Two polynomials $f, g \in \mathbb{R}[x]$ such that $f \neq g$, but $\langle f \rangle = \langle g \rangle$.

(b) Two ideals $I, J \subseteq \mathbb{R}[x]$ such that $\mathbf{V}(I) = \mathbf{V}(J)$ but $I \neq J$.

(c) An ideal $I \subseteq \mathbb{C}[x]$ such that $\mathbf{V}(I) = \{0\}$ but $I \neq \langle x \rangle$.

(d) A polynomial $f \in \mathbb{R}[x]$ such that $\mathbf{V}(f)$ consists of exactly three distinct points.

(e) A polynomial $f \in \mathbb{R}[x]$ such that $\mathbf{V}(f)$ consists of exactly three distinct points, but f has total degree 4.

(f) An ideal $I \subseteq \mathbb{R}[x, y, z]$ such that $\mathbf{V}(I)$ is infinite.

(g) An ideal $I \subseteq \mathbb{R}[x, y, z]$ such that $\mathbf{V}(I)$ is finite.

(h) A sequence of polynomials $f_1, \ldots, f_s$ in x, y such that $\mathbf{V}(f_1, \ldots, f_s)$ is a non-empty finite set when the polynomials are considered as elements of $\mathbb{R}[x, y]$, but $\mathbf{V}(f_1, \ldots, f_s)$ is infinite when the polynomials are regarded as elements of $\mathbb{C}[x, y]$.

(i) A polynomial f in $\mathbb{Q}[x, y, z]$ such that $\mathrm{LT}(f)$ is different for each of the monomial orders *lex*, *grlex*, and *grevlex*.

(j) A sequence of polynomials $f_1, \ldots, f_n$ that is *not* a Gröbner basis for $\langle f_1, \ldots, f_n \rangle$.

Problem 8 Suppose $f \in k[x_1, \ldots, x_n]$ is a polynomial. Prove that there is finite set of monomials S_f such that, if F is an s-tuple in $k[x_1, \ldots, x_n]$ and r is the remainder of f on division by F using *grlex*, then all monomials in r belong to S_f. Why can't the same be said if *lex* order is used?